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Effective field theory of slowly-moving "extreme black holes"

Yoshitaka Degura*

Graduate School of Science and Engineering, Yamaguchi University, Yoshida, Yamaguchi-shi, Yamaguchi 753-8512, Japan

Kiyoshi Shiraishi†

Graduate School of Science and Engineering, Yamaguchi University, Yoshida, Yamaguchi-shi, Yamaguchi 753-8512, Japan

and Faculty of Science, Yamaguchi University, Yoshida, Yamaguchi-shi, Yamaguchi 753-8512, Japan

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Abstract

We consider the non-relativistic effective field theory of "extreme black holes" in the Einstein-Maxwell-dilaton theory with an arbitrary dilaton coupling. We investigate finite-temperature behavior of gas of "extreme black holes" using the effective theory. The total energy of the classical many-body system is also derived.

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*Email address: `c1997@sty.cc.yamaguchi-u.ac.jp`

†Email address: `shiraish@sci.yamaguchi-u.ac.jp`

I. INTRODUCTION

Recently the properties of black holes in the Einstein-Maxwell-dilaton (EMD) theory have been studied by many authors [1,2]. We are interested in the EMD theory since it can be regarded as an effective field theory of string and/or the gravitational theory with a dimensional reduction. An "extreme black hole", which we consider in the present paper, is a kind of solitons with balanced three long-range forces (the gravitational force, the electric force, and the dilatonic force). [1] For the balance, the charge and mass of the "black hole" must satisfy the certain condition.

For the static case, the system with an arbitrary number of "extreme black holes" is known to be described by the Papapetrou-Majumdar solution [3,4] and its descendants [5,6]. In the present paper, we consider slowly-moving "extreme black holes". Since they can be regarded as point particles, we are able to describe the collective behavior of them by a scalar field. Though the similar system has been studied before [8–11], our approach is different from previous works and may give new insights of the collective phenomena of such objects. The paper is arranged as follows.

In Sec. II, we derive a theoretical model of a non-relastivistic scalar field incorporating the low-energy interaction among "extreme black holes". As an application, we examine an isothermal sphere of "extreme black holes" by using the technique of finite-temperature field theory in Sec. III. As a result, we will show that the gas of "extreme black holes" lumps at high temperature. In Sec. IV, we derive the expression for the energy of the system after eliminating the potentials by the field equations. We also obtain the total energy of the classical many-body system. Section V is devoted to conclusion.

The derivation of the effective lagrangian for the model in $(N+1)$ dimensions is exhibited in Appendix.

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[1] If a dilaton field is coupled, the extreme limit of the black hole solution has a singularity in general. Therefore, we use the "quotation marks" around the extreme black hole in this paper.

II. EFFECTIVE LAGRANGIAN

In this section, we derive an effective lagrangian for "extreme black holes" of which a typical velocity v is small. In the classical theory, the action of particles with mass m and charge e_0, coupled to the gravitational field, the electromagnetic field A_μ, and the dilaton field ϕ, is written as

$$I = -\int ds \left[m\, e^{a\phi} + e_0 A_\mu \frac{dx^\mu}{ds}\right], \tag{1}$$

where a is the coupling constant for the dilaton field. Then, the four-momentum of the particle is

$$P_\mu = me^{a\phi} g_{\mu\nu} \frac{dx^\nu}{ds} - e_0 A_\mu\,. \tag{2}$$

This four-momentum satisfies the following equation:

$$g^{\mu\nu}(P_\mu + e_0 A_\mu)(P_\nu + e_0 A_\nu) + m^2 e^{2a\phi} = 0\,. \tag{3}$$

In quantum theory, we can rewrite this equation (3) into a wave equation for a wave function φ:

$$\left[g^{\mu\nu}(P_\mu + e_0 A_\mu)(P_\nu + e_0 A_\nu) + m^2 e^{2a\phi}\right]\varphi = 0\,, \tag{4}$$

where φ possesses information on the "dynamics" of the collective behavior. So the action which yields this wave equation is

$$S_m = \int d^4x\, \sqrt{-g}\left[-\varphi^* e^{-a\phi} g^{\mu\nu}(P_\mu + e_0 A_\mu)(P_\nu + e_0 A_\nu)\varphi - m^2 e^{a\phi}\varphi^*\varphi\right], \tag{5}$$

where we regard φ as a "field" from now on. Therefore the total action including long-range interactions is

$$S = \int d^4x\, \frac{\sqrt{-g}}{16\pi}\left[R - 2\nabla_\mu\phi\nabla^\mu\phi - e^{-2a\phi}F_{\mu\nu}F^{\mu\nu}\right] + S_m\,, \tag{6}$$

where R is the scalar curvature and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The Newton constant is set to unity.

This action leads to the following field equations:

$$\nabla^2\phi + \frac{a}{2}\,e^{-2a\phi}F^2 + \,4\pi a\Big[\,e^{-a\phi}\,\varphi^*(P+e_0A)^2\varphi - e^{a\phi}m^2\,\varphi^*\varphi\Big] = 0\,, \tag{7}$$

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R = 2\,\Big[\,\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}\,g_{\mu\nu}(\nabla\phi)^2\Big] + \,e^{-2a\phi}\Big[\,2\,F^2_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}F^2\Big] \\ +16\pi\Big\{e^{-a\phi}Re\Big[\varphi^*(P_\mu+e_0A_\mu)(P_\nu+e_0A_\nu)\,\varphi \\ -\frac{1}{2}\,g_{\mu\nu}\varphi^*(P+e_0A)^2\varphi\Big] - \frac{1}{2}\,g_{\mu\nu}e^{a\phi}m^2\varphi^*\varphi\Big\}\,, \end{aligned} \tag{8}$$

$$\nabla_\mu\Big[\,e^{-2a\phi}F^{\mu\nu}\Big] = 8\pi e_0\,e^{-a\phi}\varphi^*g^{\nu\lambda}(P_\lambda+e_0A_\lambda)\,\varphi\,, \tag{9}$$

where $F^2 = F_{\mu\nu}F^{\mu\nu}$ and $F^2_{\mu\nu} = F_{\mu\lambda}F_\nu{}^\lambda$. $R_{\mu\nu}$ is the Ricci tensor.

Now, in order to consider only the lowest order in a typical velocity v of "extreme black holes", we assume the following ansätze: [2]

$$ds^2 = -U^{-2}(dt + B_i dx^i)^2 + U^2 d\boldsymbol{x}^2\ \ , \tag{10}$$

$$U(\boldsymbol{x}) = V(\boldsymbol{x})^{\frac{1}{1+a^2}}\ \ , \tag{11}$$

$$e^{-2a\phi} = V^{\frac{2a^2}{1+a^2}}\ \ , \tag{12}$$

$$A_0(\boldsymbol{x}) = \frac{1}{\sqrt{1+a^2}}\Big(1 - \frac{1}{V}\Big)\ \ , \tag{13}$$

$$A_i(\boldsymbol{x}) \sim B_i(\boldsymbol{x}) = O(v)\,, \tag{14}$$

where $i, j, \ldots$ denotes the spatial indices.

If there is no matter source and the vacuum is static ($A_i = B_i = 0$), the ansätze together with the mass-charge relation

$$\frac{e_0}{m} = \sqrt{1+a^2}\,, \tag{15}$$

reduce the field equations to

$$\partial^2 V = 0\,, \tag{16}$$

which implies that the vacuum solution represents an arbitrary number of "extreme black holes" [3–6]. One can find that the relation between mass m and electric charge e_0 of a particle corresponds to "extreme black holes". Thus we take this relation (15) hereafter.

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[2] These ansätse are the same as those in [7–11].

Since we consider the low energy limit, $-P_0 - m = E - m \ll m$, we find

$$
\begin{aligned}
P_0 + e_0 A_0 &= P_0 + \frac{e_0}{\sqrt{1+a^2}}\Big(1 - \frac{1}{V}\Big) = P_0 + m\Big(1 - \frac{1}{V}\Big) \\
&\approx -m\,\frac{1}{V}\,, \qquad (17) \\
P_i + e_0 A_i - B_i(P_0 + e_0 A_0) &\approx P_i + e_0\Big(A_i + \frac{1}{\sqrt{1+a^2}}\,\frac{1}{V}\,B_i\Big) \\
&\equiv P_i + e_0 \hat{A}_i\,, \qquad (18)
\end{aligned}
$$

where

$$
\hat{A}_i \equiv A_i + \frac{1}{\sqrt{1+a^2}}\,\frac{1}{V}\,B_i\,, \qquad (19)
$$

and we define

$$
\begin{aligned}
\hat{F}_{ij} &\equiv \partial_i \hat{A}_j - \partial_j \hat{A}_i \\
&= \bar{F}_{ij} + \frac{1}{\sqrt{1+a^2}}\,\frac{1}{V}\,G_{ij}\,, \qquad (20)
\end{aligned}
$$

where

$$
\bar{F}_{ij} \equiv F_{ij} + B_i F_{j0} - B_j F_{i0}\,, \qquad (21)
$$

$$
G_{ij} \equiv \partial_i B_j - \partial_j B_i\,. \qquad (22)
$$

Taking the low energy or non-relativistic limit $-P_0 - m = E - m \ll m$, $|P_i + e\hat{A}_i|^2 \approx m^2 v^2 \ll m^2$, we simplify the field equations (7), (8) and (9) explicitly. From the dilaton field equation (7), the time-time component of the gravitational field equation (8), and the temporal component of the electromagnetic field equation (9), in the lowest order, we obtain

$$
\partial^2 V + 8\pi(1+a^2) m^2 U^3 |\varphi|^2 = 0\,. \qquad (23)
$$

From the time-space component of the gravitational field equation (8), we get

$$
\begin{aligned}
\partial_\ell\Big[V^{\frac{2(a^2-1)}{1+a^2}}\,\frac{1}{V^2}\,G_{\ell i}\Big] = &-4\sqrt{\frac{1}{1+a^2}}\,\partial_\ell\Big[V^{\frac{2(a^2-1)}{1+a^2}}\,\frac{1}{V}\,\bar{F}_{\ell i}\Big] + 4\sqrt{\frac{1}{1+a^2}}\,\frac{1}{V}\,\partial_\ell\Big[V^{\frac{2(a^2-1)}{1+a^2}}\,\bar{F}_{\ell i}\Big] \\
&- 32\pi\,\frac{m}{V}\,e^{-a\phi}\,\varphi^*(P_i + e_0 A_i)\,\varphi\,, \qquad (24)
\end{aligned}
$$

On the other hand, the spatial component of the electromagnetic field equation (9) reads

$$\partial_\ell \Big[V^{\frac{2(a^2-1)}{1+a^2}}\, \bar{F}_{\ell i} \Big] = 8\pi e_0\, e^{-a\phi}\, \varphi^* (P_i + e_0 A_i)\, \varphi\,. \tag{25}$$

Here we define an antisymmetric tensor field H_{ij} as

$$H_{ij} \equiv 4\sqrt{\frac{1}{1+a^2}} \bar{F}_{ij} + \frac{1}{V}\, G_{ij}. \tag{26}$$

From Eqs. (24) and (25), the equation for the antisymmetric tensor field is found to be

$$\partial_\ell \Big[V^{\frac{2(a^2-1)}{1+a^2}}\, \frac{1}{V}\, H_{\ell i} \Big] = 0. \tag{27}$$

Note that from Eqs. (20) and (26), we can obtain the following relation:

$$\frac{1}{4}\frac{1}{V^2}\, G^2 - \bar{F}^2 = \frac{1+a^2}{3-a^2}\left(\hat{F}^2 - \frac{1}{4}\, H^2 \right), \tag{28}$$

where we assumed $a^2 \neq 3$.

Now we consider the effective lagrangian up to the lowest order. Taking the above estimations into the total action (6), we find:

$$S = \int d^4x \mathcal{L}\,, \tag{29}$$

where the effective lagrangian density is

$$\begin{aligned}
\mathcal{L} &= \sqrt{-g}\,\Big\{ \frac{1}{16\pi} \left[R - 2\,(\nabla\phi)^2 - e^{-2a\phi}\, F^2 \right] \\
&\qquad\qquad + \Big[-\varphi^* e^{-a\phi} g^{\mu\nu} (P_\mu + e_0 A_\mu)(P_\nu + e_0 A_\nu)\,\varphi - m^2 e^{a\phi} \varphi^* \varphi \Big] \Big\} \\
&\approx \frac{1}{16\pi}\, V^{\frac{2(a^2-1)}{1+a^2}} \left[\frac{1}{4}\frac{1}{V^2}\, G^2 - \bar{F}^2 \right] \\
&\quad + \Big[V\, U^3 \varphi^* (P_0 + e_0 A_0)^2 \varphi - \frac{1}{V} m^2 U^3 \varphi^* \varphi - \frac{1}{V}\, V^{\frac{2(a^2-1)}{1+a^2}}\, U^3 \varphi^* (P_i + e_0 \hat{A}_i)^2 \varphi \Big] \\
&= \frac{1}{16\pi}\, V^{\frac{2(a^2-1)}{1+a^2}} \left[\frac{1+a^2}{3-a^2} \left(\hat{F}^2 - \frac{1}{4}\, H^2 \right) \right] \\
&\quad + \Big[V U^3 \varphi^* (P_0 + e_0 A_0)^2 \varphi - \frac{1}{V} m^2 U^3 \varphi^* \varphi - \frac{1}{V}\, V^{\frac{2(a^2-1)}{1+a^2}}\, U^3 \varphi^* (P_i + e_0 \hat{A}_i)^2 \varphi \Big]\,.
\end{aligned} \tag{30}$$

In the first term of this effective lagrangian, H_{ij} can be regarded as an independent field, for H_{ij} does not couple to the scalar field φ. Because we consider only the interactions among black holes, we can set $H_{ij} \equiv 0$.

Then Eq. (24), which comes from the time-space component of the gravitational field equation (8), and Eq. (25), from the spatial component of the electromagnetic field equation (9), can be read as

$$-\frac{1+a^2}{3-a^2}\,\partial_\ell\Big[V^{\frac{2(a^2-1)}{1+a^2}}\,\hat{F}_{\ell i}\Big] = 8\pi e_0\, e^{-a\phi}\,\varphi^*(P_i + e_0\hat{A}_i)\,\varphi\,, \tag{31}$$

where we assumed $a^2 \neq 3$. For $a^2 = 3$, the scalar field does not couple to the force field. For a while, we take $a^2 \neq 3$.

To proceed further, we introduce a non-relativistic field ψ:

$$\psi \equiv \sqrt{2m}\,U^{3/2}\,\varphi\,, \tag{32}$$

where for the spatial measure $(g^{(3)})^{1/4} = U^{3/2}$, we obtain a correct measure for a usual spatial volume.

Finally, using the approximation

$$\begin{aligned}(P_0 + e_0 A_0)^2 &= \left(-P_0 - m + \frac{m}{V}\right)^2 \\ &\approx \frac{m^2}{V^2} + 2\frac{m}{V}\left(-P_0 - m\right)\,,\end{aligned} \tag{33}$$

together, we get the effective lagrangian density in the low energy limit:

$$\begin{aligned}\mathcal{L} = \psi^*\,(-\,P_0 - m)\psi &- \frac{1}{2m\,V^{(3-a^2)/(1+a^2)}}\,\psi^*\,(\boldsymbol{P} + e_0\hat{\boldsymbol{A}})^2\psi \\ &+ \frac{1}{16\pi}\,\frac{1+a^2}{3-a^2}\,\frac{1}{V^{2(1-a^2)/(1+a^2)}}\,\hat{F}^2 \qquad (a^2 \neq 3),\end{aligned} \tag{34}$$

where V satisfies the following equation:

$$\partial^2 V + 4\pi\,(1+a^2)\,m\,|\psi|^2 = 0\,. \tag{35}$$

As a check, varying this effective lagrangian (34) with respect to $\hat{\boldsymbol{A}}$, we can derive again the field equation (31) in the low energy approximation:

$$\begin{aligned}-\frac{1+a^2}{3-a^2}\,\partial_\ell\Big[V^{\frac{2(a^2-1)}{1+a^2}}\,\hat{F}_{\ell i}\Big] &= 4\pi\frac{e_0}{m}\,V^{\frac{a^2-3}{1+a^2}}\,\psi^*(P_i + e_0\hat{A}_i)\,\psi \\ &= 8\pi e_0\, e^{-a\phi}\,\varphi^*(P_i + e_0\hat{A}_i)\,\varphi \qquad (a^2 \neq 3)\,.\end{aligned} \tag{36}$$

For $a^2 = 3$, since the scalar field does not couple to the vector potential, the effective lagrangian density at the lowest order is

$$\mathcal{L} = \psi^* \, (-\, P_0 - m)\psi - \frac{1}{2m} \, \psi^* \, \boldsymbol{P}^2 \psi \qquad (a^2 = 3) \, , \tag{37}$$

which seems to describe a free field in the low energy limit.

III. GAS OF "EXTREME BLACK HOLES" AT FINITE TEMPERATURE

In this section, we apply the effective theory to the study of the thermal system of "extreme black holes". First we rewrite the effective lagrangian (34) derived in the preceding section as follows: [3]

$$\mathcal{L} = \psi^* \, i \frac{\partial}{\partial t} \, \psi - \frac{1}{2\tilde{m}} \, \psi^* \, (\boldsymbol{P} + \tilde{e}\hat{\boldsymbol{A}})^2 \psi + \frac{1}{16\pi} \, \hat{F}^2 \, , \tag{38}$$

where we have used the notations:

$$\tilde{m} \equiv m \, V^{\frac{3-a^2}{1+a^2}} \, , \tag{39}$$

$$\begin{aligned} \tilde{e}^2 &\equiv \frac{3-a^2}{1+a^2} \, V^{\frac{2(1-a^2)}{1+a^2}} \, e_0^2 \\ &= (3-a^2) V^{\frac{2(1-a^2)}{1+a^2}} \, m^2 \, . \end{aligned} \tag{40}$$

Here we have used the relation between mass m and electric charge e_0 is again

$$\frac{e_0}{m} = \sqrt{1+a^2} \, . \tag{41}$$

Next we consider the field theory at finite temperature [12]. In this approach, we assume that V is nearly constant and ignore the problem of operator ordering. We define the propagator of the vector field $\hat{\boldsymbol{A}}$:

$$\nu_{ij}(\boldsymbol{q}) = \nu(\boldsymbol{q}) \Big(\delta_{ij} - \frac{q_i q_j}{q^2} \Big) \, . \tag{42}$$

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[3]We assume the case of $a^2 \neq 3$, unless particularly indicated.

In the lowest order, the self-energy of the scalar field, here assumed to follow Bose-Einstein statistics, is [4]

$$\Sigma(\boldsymbol{k}) = -\frac{1}{\beta\mathcal{V}}\frac{1}{4\tilde{m}^2}\sum_{q,\ell}\frac{\nu(\boldsymbol{k}-\boldsymbol{q})}{i\omega_\ell - \epsilon'_q}\left[(\boldsymbol{k}+\boldsymbol{q})^2 - \frac{\{(\boldsymbol{k}+\boldsymbol{q})\cdot(\boldsymbol{k}-\boldsymbol{q})\}^2}{(\boldsymbol{k}-\boldsymbol{q})^2}\right], \tag{43}$$

where $\mathcal{V}$ is the volume of the system, $\beta = 1/T$ (T is the temperature of the system) and $\omega_\ell = 2\ell\pi/\beta$. ϵ'_k is considered as

$$\epsilon'_k = \frac{1}{2\tilde{m}}\boldsymbol{k}^2 + \Sigma(\boldsymbol{k}) - \mu\,, \tag{44}$$

where μ denotes the chemical potential for the scalar field. In the lowest order in interactions, $\nu(\boldsymbol{q}) = \tilde{e}^2/\boldsymbol{q}^2$. Recalling that we have assumed that the variation of the background field, or, that of V is small.

We then transform the sum over $\boldsymbol{q}$ into an integral representation:

$$\frac{1}{\mathcal{V}}\sum_{\boldsymbol{q}} \qquad \Longrightarrow \qquad \int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\,, \tag{45}$$

and the sum over ℓ in this case yields

$$\sum_\ell \frac{1}{i\omega_\ell - x} = -\frac{\beta}{e^{\beta x}-1}. \tag{46}$$

Using these, we can rewrite the self-energy of the scalar field (43) as

$$\begin{aligned}\Sigma(\boldsymbol{k}) &= \frac{\tilde{e}^2}{4\tilde{m}^2}\frac{1}{(2\pi)^3}\int d^3\boldsymbol{q}\, f_q \frac{1}{(\boldsymbol{k}-\boldsymbol{q})^2}\left[(\boldsymbol{k}+\boldsymbol{q})^2 - \frac{\{(\boldsymbol{k}+\boldsymbol{q})\cdot(\boldsymbol{k}-\boldsymbol{q})\}^2}{(\boldsymbol{k}-\boldsymbol{q})^2}\right] \\ &= \frac{\tilde{e}^2}{\tilde{m}^2}\frac{1}{4\pi^2}\int_0^\infty dq\, q^2 f_q\left[-1 - \frac{k^2+q^2}{2kq}\ln\left|\frac{k-q}{k+q}\right|\right],\end{aligned} \tag{47}$$

where f_q is a distribution function of the scalar field:

$$f_q \equiv \frac{1}{e^{\beta\epsilon'_q}-1}\,. \tag{48}$$

For a small k, corresponding to the low energy, we find:

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[4]There is no tadopole contribution, because of the derivative coupling.

$$-1-\frac{k^2+q^2}{2kq}\ln\left|\frac{k-q}{k+q}\right| \approx \frac{4k^2}{3q^2}\,. \tag{49}$$

Since there is no constant term in terms of k, we can write the self-energy approximately as

$$\Sigma(\boldsymbol{k}) \approx \frac{1}{2}Bk^2\,, \tag{50}$$

where B is a constant.

From now on we consider the high-temperature approximation. [5] Then the distribution function of the scalar field is reduced to

$$f_q \approx e^{-\beta\epsilon'_q}\,. \tag{51}$$

Therefore the self-consistent equation is expressed as

$$\begin{aligned}\Sigma(\boldsymbol{k}) &\approx \frac{1}{2}Bk^2\\ &\approx \frac{\tilde{e}^2}{4\pi^2\tilde{m}^2}\int_0^\infty dq\,q^2 f_q\,\frac{4k^2}{3q^2}\\ &= \frac{1}{2}\,\frac{4\,\tilde{e}^2\,e^{\beta\mu}}{3(2\pi)^{3/2}\tilde{m}^2}\left[\beta(\frac{1}{\tilde{m}}+B)\right]^{-1/2}k^2\,.\end{aligned} \tag{52}$$

Equivalently, we get an equation for the coefficient B:

$$B = \frac{4\,\tilde{e}^2\,e^{\beta\mu}}{3(2\pi)^{3/2}\tilde{m}^2}\left[\beta(\frac{1}{\tilde{m}}+B)\right]^{-1/2}. \tag{53}$$

The solution of Eq. (53) is

$$B = \frac{4\,\tilde{e}^2\,e^{\beta\mu}}{3(2\pi)^{3/2}\tilde{m}^2}\sqrt{\frac{\tilde{m}}{\beta}}\left(\cos\frac{\theta}{3}+\frac{1}{\sqrt{3}}\sin\frac{\theta}{3}\right)^{-1}, \tag{54}$$

where

$$\theta = \arcsin\frac{2\sqrt{3}\,\tilde{e}^2 e^{\beta\mu}}{(2\pi)^{3/2}\sqrt{\beta\tilde{m}}}\,. \tag{55}$$

On the other hand, the particle density ρ is expressed as

[5] The particles are still non-relativistic and obey the Maxwell-Boltzmann distribution.

$$\rho = \frac{1}{(2\pi)^3}\int d^3\boldsymbol{q}\, f_q$$
$$= \frac{e^{\beta\mu}}{(2\pi)^{3/2}}\Big[\beta(\frac{1}{\tilde{m}}+B)\Big]^{-3/2}. \qquad (56)$$

In our model the particle density ρ can also be written by $|\psi|^2$, therefore

$$|\psi|^2 = \rho = e^{\beta\mu}\Big(\frac{\tilde{m}}{2\pi\beta}\Big)^{3/2}\Big[\cos\frac{\theta}{3}+\frac{1}{\sqrt{3}}\sin\frac{\theta}{3}\Big]^{-3}. \qquad (57)$$

Now we have to solve the following equation with $|\psi|^2$ given by Eq. (57)

$$\partial^2 V + 4\pi\,(1+a^2)\,m\,|\psi|^2 = 0\,. \qquad (58)$$

To explicitly solve Eq. (58), let us assume spherical symmetry and define the following parameters and the radial coordinate:

$$\rho_0 \equiv e^{\beta\mu}\Big(\frac{m}{2\pi\beta}\Big)^{3/2}, \qquad (59)$$
$$\delta \equiv 2\sqrt{3}\,G\,\rho_0\beta\,, \qquad (60)$$
$$\tilde{r} \equiv \sqrt{G\,m\,\rho_0}\,r\,, \qquad (61)$$

where G is the Newton constant, being set to unity until now. Incidentally, n_0 is the particle density of the ideal gas of particles with mass m. Then Eq. (58) for V is

$$\frac{1}{\tilde{r}^2}\Big(\tilde{r}^2 V(\tilde{r})'\Big)' = -4\pi\,(1+a^2)V(\tilde{r})^{\frac{3(3-a^2)}{2(1+a^2)}}\Big(\cos\frac{\theta}{3}+\frac{1}{\sqrt{3}}\sin\frac{\theta}{3}\Big)^{-3}, \qquad (62)$$

where the prime denotes derivative with respect to $\tilde{r}$, and

$$\theta(\tilde{r}) \;=\; \arcsin\Big[(3-a^2)V(\tilde{r})^{\frac{(1-3a^2)}{2(1+a^2)}}\,\delta\Big]. \qquad (63)$$

We can numerically solve Eq. (62) with Eq. (63) to obtain $V(\tilde{r})$ and the particle density $\rho = |\psi|^2$ of the isothermal sphere of "extreme black holes". Fig. 1 shows the profile of the particle density of the isothermal sphere. The central density is normalized to unity there. As seen from Fig. 1, as δ is the smaller, that is, the temperature is the higher, the particles are bound the more tightly.

We find also that as the coupling constant of the dilaton field is the larger, the particles are bound the more tightly. In particular, there is a critical value for the asymptotic behavior of n at a large distance: for $a^2 > 1$, the isothermal sphere at high temperature has a clear "edge".

For the case $a^2 = 3$, the effective lagrangian represents the free lagrangian, therefore the particles correspond to the free particles.

The physical interpretation on the behavior is given by the fact that the "magnetic" force acts attractively between the particles which move in the opposite direction for $a^2 < 3$ (see the relative sign of the coefficient in Eq. (36)). The thermal average of v^2 governs the strength of the force; this feature has been suggested in [13].

We have also seen that there is the critical point at $a^2 \sim 1$ whether an extent of the "edge" exists or not. This behavior implies the existence of an effective repulsion in a small range, because similar behavior can be found in the model of boson stars with a repulsive force [14]. This critical nature of the interaction seems to correspond to the fact that the moduli space of two "extreme black holes" has a deficit angle π for $a^2 = 1$ [11]. Further study on this subject is expected.

IV. ELIMINATING THE POTENTIALS

We consider the total energy of the system of "extreme black holes". We can eliminate the field stength and the potential apparently in the expression of the energy using the solution of the field equations expressed by the scalar field. Assuming localized scalar distributions in the resulting expression, we can also obtain the energy for the classical system. For a while, we assume $a^2 \neq 3$.

The total energy of the system is written by

$$H = \int d^3\boldsymbol{x}\ \mathcal{H}\,, \tag{64}$$

with

$$\mathcal{H} = \frac{1}{2m\,V^{(3-a^2)/(1+a^2)}}\,\psi^*\,(\boldsymbol{P}+e_0\hat{\boldsymbol{A}})^2\psi - \frac{1}{16\pi}\,\frac{1+a^2}{3-a^2}\,\frac{1}{V^{2(1-a^2)/(1+a^2)}}\,\hat{F}^2\,, \tag{65}$$

where $e_0 = \sqrt{1+a^2}\,m$.

First, we treat the second term of the Hamiltonian density $\mathcal{H}$. We use the following definition:

$$\begin{aligned}\boldsymbol{J}(\boldsymbol{x}) = \frac{\partial\mathcal{H}}{\partial\boldsymbol{P}} &= \frac{1}{m\,V^{(3-a^2)/(1+a^2)}}\,\psi^*\,(\boldsymbol{P}+e_0\hat{\boldsymbol{A}})\psi \\ &\equiv \psi^*(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x})\psi(\boldsymbol{x})\,.\end{aligned} \tag{66}$$

Then the field equation (36) can be written as

$$-\frac{1+a^2}{3-a^2}\,\partial_\ell\Big[V^{\frac{2(a^2-1)}{1+a^2}}\,\hat{F}_{\ell i}\Big] = 4\pi e_0\,\boldsymbol{J}\,. \tag{67}$$

Note that

$$\boldsymbol{\nabla}\cdot\boldsymbol{J} = \partial_i\,J^i = 0\,. \tag{68}$$

As long as $\boldsymbol{J}$ vanishes rapidly at the spatial infinity, the solution is given by

$$-\frac{1+a^2}{3-a^2}V^{\frac{2(a^2-1)}{1+a^2}}\,\hat{F}_{\ell i} = e_0\int d^3\boldsymbol{x}'\,\frac{r'^{\ell}J^i(\boldsymbol{x}') - r'^{i}J^{\ell}(\boldsymbol{x}')}{|\boldsymbol{r}'|^3}\,, \tag{69}$$

where $\boldsymbol{r}' = \boldsymbol{x}-\boldsymbol{x}'$. Substituting this solution, we obtain

$$\begin{aligned}&\int d^3\boldsymbol{x}\frac{1}{16\pi}\,\frac{1+a^2}{3-a^2}\,\frac{1}{V^{2(1-a^2)/(1+a^2)}}\,\hat{F}^2 \\ &= \frac{1}{8\pi}\frac{3-a^2}{1+a^2}e_0^2\int d^3\boldsymbol{x}\,V^{\frac{2(1-a^2)}{1+a^2}}(\boldsymbol{x}) \\ &\quad\times\int d^3\boldsymbol{x}'\int d^3\boldsymbol{x}''\frac{\boldsymbol{r}'\cdot\boldsymbol{r}''\;\boldsymbol{J}(\boldsymbol{x}')\cdot\boldsymbol{J}(\boldsymbol{x}'') - \boldsymbol{r}'\cdot\boldsymbol{J}(\boldsymbol{x}'')\;\boldsymbol{r}''\cdot\boldsymbol{J}(\boldsymbol{x}')}{|\boldsymbol{r}'|^3|\boldsymbol{r}''|^3}\,,\end{aligned} \tag{70}$$

where $\boldsymbol{r}'' = \boldsymbol{x}-\boldsymbol{x}''$.

To garantee the absence of the influence of the spatial infinity explicitly, we use the following identity which holds if $\boldsymbol{J}$ vanishes rapidly at the spatial infinity:

$$\int d^3\boldsymbol{x}'\;\frac{\boldsymbol{r}'\cdot\boldsymbol{J}(\boldsymbol{x}')}{|\boldsymbol{r}'|^3} = 0\,, \tag{71}$$

and modify the equation (70) into

$$
\begin{aligned}
&\int d^3\boldsymbol{x}\ \frac{1}{16\pi}\frac{1+a^2}{3-a^2}\frac{1}{V^{2(1-a^2)/(1+a^2)}}\,\hat{F}^2 \\
&= \frac{1}{8\pi}\frac{3-a^2}{1+a^2}e_0^2\int d^3\boldsymbol{x}\,V^{\frac{2(1-a^2)}{1+a^2}}(\boldsymbol{x}) \\
&\quad\times\int d^3\boldsymbol{x}'\int d^3\boldsymbol{x}''\ \frac{\boldsymbol{r}'\cdot\boldsymbol{r}''\ \boldsymbol{J}(\boldsymbol{x}')\cdot\boldsymbol{J}(\boldsymbol{x}'')-\boldsymbol{r}'\cdot\boldsymbol{J}(\boldsymbol{x}'')\ \boldsymbol{r}''\cdot\boldsymbol{J}(\boldsymbol{x}')+\boldsymbol{r}'\cdot\boldsymbol{J}(\boldsymbol{x}')\ \boldsymbol{r}''\cdot\boldsymbol{J}(\boldsymbol{x}'')}{|\boldsymbol{r}'|^3|\boldsymbol{r}''|^3} \\
&= \frac{1}{8\pi}\frac{3-a^2}{1+a^2}e_0^2\int d^3\boldsymbol{x}\,V^{\frac{2(1-a^2)}{1+a^2}}(\boldsymbol{x}) \\
&\qquad\qquad\times\int d^3\boldsymbol{x}'\int d^3\boldsymbol{x}''\ \frac{{r'}^i{r''}^j}{|\boldsymbol{r}'|^3|\boldsymbol{r}''|^3}\left(\delta_{k\ell}\delta_{ij}-\delta_{i\ell}\delta_{jk}+\delta_{ik}\delta_{j\ell}\right)J^k(\boldsymbol{x}')J^\ell(\boldsymbol{x}'')\,. \qquad (72)
\end{aligned}
$$

Next, we consider the first term of the Hamiltonian density $\mathcal{H}$. If we define

$$\rho(\boldsymbol{x})=\psi^*(\boldsymbol{x})\psi(\boldsymbol{x})\,, \qquad (73)$$

then the equation (35), which is now written as

$$\partial^2 V+4\pi\,(1+a^2)\,m\,\rho=0\,, \qquad (74)$$

has the solution:

$$V(\boldsymbol{x})=1+(1+a^2)\,m\int d^3\boldsymbol{x}'\frac{1}{|\boldsymbol{r}'|}\rho(\boldsymbol{x}')\,. \qquad (75)$$

We rearrange the first term of $\mathcal{H}$ as

$$
\begin{aligned}
\frac{1}{2m\,V^{(3-a^2)/(1+a^2)}}\,\psi^*\,(\boldsymbol{P}+e_0\hat{\boldsymbol{A}})^2\psi &= \frac{1}{2}\,m\,V^{(3-a^2)/(1+a^2)}\,\psi^*\boldsymbol{v}^2\psi \\
&= \frac{1}{2}\,m\,\psi^*\boldsymbol{v}^2\psi+\frac{1}{2}m\,[V^{\frac{3-a^2}{1+a^2}}-1]\,\psi^*\boldsymbol{v}^2\psi\,. \qquad (76)
\end{aligned}
$$

Using Eq (75), we find

$$
\begin{aligned}
&\frac{1}{2}\,m\int d^3\boldsymbol{x}\ [V^{\frac{3-a^2}{1+a^2}}(\boldsymbol{x})-1]\,\psi^*(\boldsymbol{x})\boldsymbol{v}^2(\boldsymbol{x})\psi(\boldsymbol{x}) \\
&= \frac{1}{8\pi}\,m\int d^3\boldsymbol{x}\ [V^{\frac{3-a^2}{1+a^2}}(\boldsymbol{x})-1]\int d^3\boldsymbol{x}'\ \boldsymbol{\nabla}\cdot\frac{\boldsymbol{r}'}{|\boldsymbol{r}'|^3}\psi^*(\boldsymbol{x}')\boldsymbol{v}^2(\boldsymbol{x}')\psi(\boldsymbol{x}') \\
&= \frac{3-a^2}{8\pi}\,m^2\int d^3\boldsymbol{x}\,V^{\frac{2(1-a^2)}{1+a^2}}(\boldsymbol{x}) \\
&\qquad\qquad\times\int d^3\boldsymbol{x}'\int d^3\boldsymbol{x}''\,\frac{\boldsymbol{r}'\cdot\boldsymbol{r}''}{|\boldsymbol{r}'|^3|\boldsymbol{r}''|^3}\psi^*(\boldsymbol{x}'')\psi(\boldsymbol{x}'')\psi^*(\boldsymbol{x}')\boldsymbol{v}^2(\boldsymbol{x}')\psi(\boldsymbol{x}')\,. \qquad (77)
\end{aligned}
$$

Consequently, the interaction energy can be written as

$$\int d^3\boldsymbol{x}\,\Big\{\frac{1}{2}\,m\,[V^{\frac{3-a^2}{1+a^2}}(\boldsymbol{x})-1]\,\psi^*(\boldsymbol{x})\boldsymbol{v}^2(\boldsymbol{x})\psi(\boldsymbol{x})-\frac{1}{16\pi}\frac{1+a^2}{3-a^2}\frac{1}{V^{2(1-a^2)/(1+a^2)}}\hat{F}^2\Big\}$$
$$=\frac{3-a^2}{8\pi}\,m^2\int d^3\boldsymbol{x}\,V^{\frac{2(1-a^2)}{1+a^2}}(\boldsymbol{x})\int d^3\boldsymbol{x}'\int d^3\boldsymbol{x}''\,\frac{1}{|\boldsymbol{r}'|^3|\boldsymbol{r}''|^3}$$
$$\times\psi^*(\boldsymbol{x}')\psi^*(\boldsymbol{x}'')\Big\{\frac{1}{2}\boldsymbol{r}'\cdot\boldsymbol{r}''|\boldsymbol{v}(\boldsymbol{x}')-\boldsymbol{v}(\boldsymbol{x}'')|^2-(\boldsymbol{r}'\times\boldsymbol{r}'')\cdot(\boldsymbol{v}(\boldsymbol{x}')\times\boldsymbol{v}(\boldsymbol{x}''))\Big\}\psi(\boldsymbol{x}')\psi(\boldsymbol{x}'')\,. \tag{78}$$

Because we know that there is no interaction for the $a^2=3$ case, we see that this expression holds for any value of the dilaton coupling a^2.

For $a^2=1$, the expression for the total enegy can be simplified into

$$\begin{aligned}H=&\int d^3\boldsymbol{x}\,\frac{1}{2}\,m\,\psi^*(\boldsymbol{x})\boldsymbol{v}^2(\boldsymbol{x})\psi(\boldsymbol{x})\\&+\frac{1}{8\pi}\,m^2\int d^3\boldsymbol{x}\int d^3\boldsymbol{x}'\int d^3\boldsymbol{x}''\,\frac{\boldsymbol{r}'\cdot\boldsymbol{r}''}{|\boldsymbol{r}'|^3|\boldsymbol{r}''|^3}\psi^*(\boldsymbol{x}')\psi^*(\boldsymbol{x}'')|\boldsymbol{v}(\boldsymbol{x}')-\boldsymbol{v}(\boldsymbol{x}'')|^2\psi(\boldsymbol{x}')\psi(\boldsymbol{x}'')\\=&\int d^3\boldsymbol{x}\,\frac{1}{2}\,m\,\psi^*(\boldsymbol{x})\boldsymbol{v}^2(\boldsymbol{x})\psi(\boldsymbol{x})\\&+\frac{1}{2}\,m^2\int d^3\boldsymbol{x}\int d^3\boldsymbol{x}'\,\psi^*(\boldsymbol{x})\psi^*(\boldsymbol{x}')\,\frac{|\boldsymbol{v}(\boldsymbol{x})-\boldsymbol{v}(\boldsymbol{x}')|^2}{|\boldsymbol{x}-\boldsymbol{x}'|}\,\psi(\boldsymbol{x})\psi(\boldsymbol{x}')\,.\end{aligned} \tag{79}$$

Now we consider the classical point-particle limit,

$$\boldsymbol{J}(\boldsymbol{x})=\psi^*(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x})\psi(\boldsymbol{x})=\sum_a\mu_a\boldsymbol{v}_a\,\delta^3(\boldsymbol{r}_a)\,, \tag{80}$$
$$\rho(\boldsymbol{x})=\psi^*(\boldsymbol{x})\psi(\boldsymbol{x})=\sum_a\mu_a\,\delta^3(\boldsymbol{r}_a)\,, \tag{81}$$

and

$$\psi^*(\boldsymbol{x})\boldsymbol{v}^2(\boldsymbol{x})\psi(\boldsymbol{x})=\sum_a\mu_a\boldsymbol{v}_a^2\,\delta^3(\boldsymbol{r}_a)\,, \tag{82}$$

where μ_a and $\boldsymbol{v}_a$ are the constant which represents the ratio of the mass and the velocity of the a-th "extreme black hole" located at $\boldsymbol{x}_a$, respectively. We use the notation $\boldsymbol{r}_a=\boldsymbol{x}-\boldsymbol{x}_a$. Then one can find that the energy of the classical system takes the form:

$$\begin{aligned}H=&\sum_a\frac{1}{2}m_a\,\boldsymbol{v}_a^2\\&+\frac{3-a^2}{8\pi}\int d^3\boldsymbol{x}\,V^{\frac{2(1-a^2)}{1+a^2}}(\boldsymbol{x})\sum_{ab}\frac{m_a m_b}{|\boldsymbol{r}_a|^3|\boldsymbol{r}_b|^3}\\&\quad\times\Big\{\frac{1}{2}\boldsymbol{r}_a\cdot\boldsymbol{r}_b|\boldsymbol{v}_a-\boldsymbol{v}_b|^2-(\boldsymbol{r}_a\times\boldsymbol{r}_b)\cdot(\boldsymbol{v}_a\times\boldsymbol{v}_b)\Big\}\,,\end{aligned} \tag{83}$$

with

$$V(\boldsymbol{x}) = 1 + (1+a^2)\sum_c \frac{m_c}{|\boldsymbol{r}_c|}\,, \tag{84}$$

where the individual mass is $m_a \equiv m\mu_a$. For $a^2 = 1$, the energy has a simple form:

$$H = \sum_a \frac{1}{2}\, m_a \boldsymbol{v}_a^2 + \frac{1}{2}\sum_{ab}\, m_a m_b\, \frac{|\boldsymbol{v}_{ab}|^2}{|\boldsymbol{r}_{ab}|}\,, \tag{85}$$

where $\boldsymbol{r}_{ab} \equiv \boldsymbol{x}_a - \boldsymbol{x}_b$ and $\boldsymbol{v}_{ab} \equiv \boldsymbol{v}_a - \boldsymbol{v}_b$.

Furthermore, we restrict ourselves on the two-body system. We assume that the velocity of the center of mass $\mathbf{V}$ vanishes:

$$\mathbf{V} \equiv \frac{m_1 \boldsymbol{v}_1 + m_2 \boldsymbol{v}_2}{M} = \mathbf{0}\,, \tag{86}$$

where $M = m_1 + m_2$. The velocity of the relative motion is defined as

$$\mathbf{v} \equiv \boldsymbol{v}_1 - \boldsymbol{v}_2\,. \tag{87}$$

Then Eq. (83) becomes

$$H = \frac{1}{2}\,\mu\,\mathbf{v}^2 + \frac{3-a^2}{8\pi}\,\mu M\,\mathbf{v}^2 \int d^3\boldsymbol{x}\; V^{\frac{2(1-a^2)}{1+a^2}}(\boldsymbol{x})\,\frac{\boldsymbol{r}_1\cdot\boldsymbol{r}_2}{|\boldsymbol{r}_1|^3|\boldsymbol{r}_2|^3}\,, \tag{88}$$

with

$$V(\boldsymbol{x}) = 1 + (1+a^2)\Big[\frac{m_1}{|\boldsymbol{r}_1|} + \frac{m_2}{|\boldsymbol{r}_2|}\Big]\,, \tag{89}$$

where the reduced mass μ is given by $\mu = m_1 m_2/M$.

In general, a naive integration in Eq. (88) diverges. Therefore we realize that divergent terms proportional to $\int d^3\boldsymbol{x}\;\delta^3(\boldsymbol{x})/|\boldsymbol{x}|^p \quad (p>0)$ which appear when the integrand is expanded must be regularized [15]. We set them to zero. The prescription is equivalent to carrying out the following replacement in Eq. (88):

$$V^{\frac{2(1-a^2)}{1+a^2}}(\boldsymbol{x}) \Rightarrow \Big[1 + (1+a^2)\frac{m_1}{|\boldsymbol{r}_1|}\Big]^{\frac{2(1-a^2)}{1+a^2}} + \Big[1 + (1+a^2)\frac{m_2}{|\boldsymbol{r}_2|}\Big]^{\frac{2(1-a^2)}{1+a^2}} - 1\,. \tag{90}$$

Then we get [11]

$$H = \frac{1}{2}\,\mu\,\mathbf{v}^2\,\Big\{1 - \frac{M}{\mu} - \frac{(3-a^2)M}{r}$$
$$+\frac{M}{m_1}\left[1+(1+a^2)\frac{m_1}{r}\right]^{\frac{3-a^2}{1+a^2}} + \frac{M}{m_2}\left[1+(1+a^2)\frac{m_2}{r}\right]^{\frac{3-a^2}{1+a^2}}\Big\}\,, \tag{91}$$

where $r = |\boldsymbol{x}_1 - \boldsymbol{x}_2|$.

Before closing this section, we show the results for $(N+1)$ dimensional case. We assume $a^2 \neq N$ here. The basic action is given in Appendix. We find that the effective action is

$$\mathcal{H} = \frac{1}{2m\,V^{\frac{N-a^2}{N-2+a^2}}}\,\psi^*\,(\boldsymbol{P}+e_0\hat{\boldsymbol{A}})^2\psi - \frac{1}{16\pi}\,\frac{N-2+a^2}{N-a^2}\,\frac{1}{V^{\frac{2(1-a^2)}{N-2+a^2}}}\,\hat{F}^2\,, \tag{92}$$

where

$$e_0^2 = \frac{2(N-2+a^2)}{N-1}\,m^2\,. \tag{93}$$

Here V satisfies

$$\partial^2 V + 8\pi\,\frac{N-2+a^2}{N-1}\,m\,|\psi|^2 = 0\,. \tag{94}$$

The derivation of the effective action is shown in Appendix.

Similarly to the previous analysis, we obtain the interaction Hamiltonian:

$$H_{int} = \int d^N\boldsymbol{x}\,\Big\{\frac{1}{2}m\,[V^{\frac{N-a^2}{N-2+a^2}}(\boldsymbol{x})-1]\,\psi^*(\boldsymbol{x})\boldsymbol{v}^2(\boldsymbol{x})\psi(\boldsymbol{x})$$
$$-\frac{1}{16\pi}\,\frac{N-2+a^2}{N-a^2}\,\frac{1}{V^{2(1-a^2)/(N-2+a^2)}}\,\hat{F}^2\Big\}$$
$$= \frac{N-a^2}{4(N-1)\pi}\left(\frac{4\pi}{A_{N-1}}\right)^2 m^2\int d^N\boldsymbol{x}\;V^{\frac{2(1-a^2)}{N-2+a^2}}(\boldsymbol{x})\int d^N\boldsymbol{x}'\int d^N\boldsymbol{x}''\;\frac{1}{|\boldsymbol{r}'|^N|\boldsymbol{r}''|^N}$$
$$\times\psi^*(\boldsymbol{x}')\psi^*(\boldsymbol{x}'')\Big\{\frac{1}{2}\boldsymbol{r}'\cdot\boldsymbol{r}''|\boldsymbol{v}(\boldsymbol{x}')-\boldsymbol{v}(\boldsymbol{x}'')|^2-(\boldsymbol{r}'\cdot\boldsymbol{v}(\boldsymbol{x}'))(\boldsymbol{r}''\cdot\boldsymbol{v}(\boldsymbol{x}''))$$
$$+(\boldsymbol{r}'\cdot\boldsymbol{v}(\boldsymbol{x}''))(\boldsymbol{r}''\cdot\boldsymbol{v}(\boldsymbol{x}'))\Big\}\psi(\boldsymbol{x}')\psi(\boldsymbol{x}'')\,, \tag{95}$$

where

$$\boldsymbol{v}(\boldsymbol{x}) \equiv \frac{1}{m\,V^{(N-a^2)/(N-2+a^2)}}\,(\boldsymbol{P}+e_0\hat{\boldsymbol{A}})\,, \tag{96}$$

and $A_{N-1} = 2\pi^N/\Gamma(N/2)$ is the volume of a unit $N-1$ sphere.

Especially for $a^2 = 1$, we find

$$H_{int} = \frac{1}{2}\frac{4\pi}{A_{N-1}} m^2 \int d^N\boldsymbol{x} \int d^N\boldsymbol{x}'\ \psi^*(\boldsymbol{x})\psi^*(\boldsymbol{x}')\frac{|\boldsymbol{v}(\boldsymbol{x})-\boldsymbol{v}(\boldsymbol{x}')|^2}{(N-2)|\boldsymbol{x}-\boldsymbol{x}'|^{N-2}}\psi(\boldsymbol{x})\psi(\boldsymbol{x}')\,. \tag{97}$$

The classical limit of total energy of the system is found to be

$$\begin{aligned} H = & \sum_a \frac{1}{2} m_a\, \boldsymbol{v}_a^2 \\ & + \frac{N-a^2}{4(N-1)\pi}\left(\frac{4\pi}{A_{N-1}}\right)^2 \int d^N\boldsymbol{x}\ V^{\frac{2(1-a^2)}{N-2+a^2}}(\boldsymbol{x}) \sum_{ab} \frac{m_a m_b}{|\boldsymbol{r}_a|^N |\boldsymbol{r}_b|^N} \\ & \times\Big\{\frac{1}{2}\boldsymbol{r}_a\cdot\boldsymbol{r}_b|\boldsymbol{v}_a-\boldsymbol{v}_b|^2 - (\boldsymbol{r}_a\cdot\boldsymbol{v}_a)(\boldsymbol{r}_b\cdot\boldsymbol{v}_b) + (\boldsymbol{r}_a\cdot\boldsymbol{v}_b)(\boldsymbol{r}_b\cdot\boldsymbol{v}_a)\Big\}\,, \end{aligned} \tag{98}$$

with

$$V(\boldsymbol{x}) = 1 + \frac{2(N-2+a^2)}{(N-1)(N-2)}\frac{4\pi}{A_{N-1}}\sum_c \frac{m_c}{|\boldsymbol{r}_c|^{N-2}}\,. \tag{99}$$

For $a^2 = 1$, the total energy of the system has a simple form:

$$H = \sum_a \frac{1}{2} m_a \boldsymbol{v}_a^2 + \frac{1}{2}\frac{4\pi}{A_{N-1}}\sum_{ab} m_a m_b \frac{|\boldsymbol{v}_{ab}|^2}{(N-2)|\boldsymbol{r}_{ab}|^{N-2}}\,. \tag{100}$$

Finally, we show another expression for the total energy. That is

$$H = \sum_{ab} v^{ak} v^{b\ell}(\delta^i_k\delta^j_\ell + \delta_{k\ell}\delta^{ij} - \delta^j_k\delta^i_\ell)\partial_{ai}\partial_{bj} L\,, \tag{101}$$

where

$$L = -\frac{1}{32\pi}\int d^N\boldsymbol{x}\ V^{\frac{2(N-1)}{N-2+a^2}}(\boldsymbol{x})\,, \tag{102}$$

with V given by Eq. (99).

V. CONCLUSION

In this paper, we have derived the effective lagrangian of "extreme black holes" in the low energy limit. At finite temperature, we have obtained a self-consistent equation and then we have seen the structure of the isothermal sphere distribution of "extreme black holes". At high temperature, the gas of "extreme black holes" have been lumped by the velocity-dependent force.

In future work, we will consider the low-temperature case. It is interesting to investigate whether the condensation of “extreme black holes” may take place. We will also take another posibility for the statistics of “extreme black holes” into consideration. We would like to study the effective theory by means of the lattice calculation, in which the strongly coupled system is appropriately treated.

We have also studied the classical point-particle limit for the energy of the system. Moduli space structure and supersymmetric extensions of the multiple black hole system have recently been studied by many authors [16–26]. We are interested in such a direction of study and expect some symmetric structure to be found in the effective field theory of multi-black holes.

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APPENDIX

In this Appendix, we show the derivation of the effective lagrangian for "extreme black holes" in the $(N+1)$ dimensional spacetime in detail.

We start with the action for the charged scalar field:

$$S_m = \int d^{N+1}x \sqrt{-g} \left[-\varphi^* e^{-\frac{2a}{N-1}\phi} g^{\mu\nu}(P_\mu + e_0 A_\mu)(P_\nu + e_0 A_\nu)\varphi - m^2 e^{\frac{2a}{N-1}\phi} \varphi^* \varphi \right]. \tag{A1}$$

The total action reads

$$S = \int d^{N+1}x \frac{\sqrt{-g}}{16\pi} \left[R - \frac{4}{N-1} \nabla_\mu \phi \nabla^\mu \phi - e^{-\frac{4a}{N-1}\phi} F_{\mu\nu} F^{\mu\nu} \right] + S_m, \tag{A2}$$

and leads to the field equations:

$$\nabla^2 \phi + \frac{a}{2} e^{-\frac{4a}{N-1}\phi} F^2 + 4\pi a \Big[e^{-\frac{2a}{N-1}\phi} \varphi^* (P + e_0 A)^2 \varphi - e^{\frac{2a}{N-1}\phi} m^2 \varphi^* \varphi \Big] = 0\,, \tag{A3}$$

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{4}{N-1} \Big[\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} (\nabla\phi)^2 \Big] + e^{-\frac{4a}{N-1}\phi} \Big[2 F_{\mu\nu}^2 - \frac{1}{2} g_{\mu\nu} F^2 \Big] \\ +16\pi \Big\{ e^{-\frac{2a}{N-1}\phi} Re \Big[\varphi^* (P_\mu + e_0 A_\mu)(P_\nu + e_0 A_\nu)\, \varphi \\ -\frac{1}{2} g_{\mu\nu} \varphi^* (P + e_0 A)^2 \varphi \Big] - \frac{1}{2} g_{\mu\nu} e^{\frac{2a}{N-1}\phi} m^2 \varphi^* \varphi \Big\}, \end{aligned} \tag{A4}$$

$$\nabla_\mu \Big[e^{-\frac{4a}{N-1}\phi} F^{\mu\nu} \Big] = 8\pi e_0\, e^{-\frac{2a}{N-1}\phi} \varphi^* g^{\nu\lambda} (P_\lambda + e_0 A_\lambda)\, \varphi\,. \tag{A5}$$

The ansätze in the $(N+1)$ dimensional case are now:

$$ds^2 = -U^{-2}(dt + B_i dx^i)^2 + U^{\frac{2}{N-2}} d\boldsymbol{x}^2\ , \tag{A6}$$

$$U(\boldsymbol{x}) = V(\boldsymbol{x})^{\frac{N-2}{N-2+a^2}}\ , \tag{A7}$$

$$e^{-\frac{4a}{N-1}\phi} = V^{\frac{2a^2}{N-2+a^2}}\ , \tag{A8}$$

$$A_0(\boldsymbol{x}) = \sqrt{\frac{N-1}{2(N-2+a^2)}} \Big(1 - \frac{1}{V}\Big)\ , \tag{A9}$$

$$A_i(\boldsymbol{x}) \sim B_i(\boldsymbol{x}) = O(v)\,. \tag{A10}$$

In addition, the following charge-mass ratio is assumed:

$$\frac{e_0}{m} = \sqrt{\frac{2(N-2+a^2)}{N-1}}. \tag{A11}$$

This corresponds to that of the "extreme black holes".

Now we consider the low energy limit, $-P_0 - m = E - m \ll m$. Then

$$P_0 + e_0 A_0 = P_0 + e_0 \sqrt{\frac{N-1}{2(N-2+a^2)}} \Big(1 - \frac{1}{V}\Big) = P_0 + m\Big(1 - \frac{1}{V}\Big) \approx -m\,\frac{1}{V}\,, \tag{A12}$$

$$P_i + e_0 A_i - B_i(P_0 + e_0 A_0) \approx P_i + e_0 \left(A_i + \sqrt{\frac{N-1}{2(N-2+a^2)}}\,\frac{1}{V}\,B_i\right) \equiv P_i + e_0 \hat{A}_i\,, \tag{A13}$$

where

$$\hat{A}_i \equiv A_i + \sqrt{\frac{N-1}{2(N-2+a^2)}}\,\frac{1}{V}\,B_i\,, \tag{A14}$$

and we define

$$\hat{F}_{ij} \equiv \partial_i \hat{A}_j - \partial_j \hat{A}_i = \bar{F}_{ij} + \sqrt{\frac{N-1}{2(N-2+a^2)}}\,\frac{1}{V}\,G_{ij}\,, \tag{A15}$$

where

$$\bar{F}_{ij} \equiv F_{ij} + B_i F_{j0} - B_j F_{i0}\,, \tag{A16}$$

$$G_{ij} \equiv \partial_i B_j - \partial_j B_i\,. \tag{A17}$$

Using the ansätze and taking the low energy or non-relativistic limit $-P_0 - m = E - m \ll m$, $|P_i + e\hat{A}_i|^2 \approx m^2 v^2 \ll m^2$, we simplify the field equations.

We reduce the dilaton field equation (A3), using Eqs. (A6), (A13), (A14), and (A16), to

$$\begin{aligned} &U^{-\frac{2}{N-2}} \partial^2 \phi + \frac{a}{2}\, e^{-\frac{4a}{N-1}\phi} \Big[-2U^2 U^{-\frac{2}{N-2}} (F_{0i})^2 + U^{-\frac{4}{N-2}} \bar{F}^2\Big] \\ &+ 4\pi a \Big\{ e^{-\frac{2a}{N-1}\phi} \left[-U^2 \varphi^* (P_0 + e_0 A_0)^2 \varphi + U^{-\frac{2}{N-2}} \varphi^* (P_i + e_0 \hat{A}_i)^2 \varphi\right] \\ &\qquad\qquad\qquad - e^{\frac{2a}{N-1}\phi} m^2\, \varphi^* \varphi \Big\} = 0\,, \end{aligned} \tag{A18}$$

where $\bar{F}^2 = \bar{F}_{ij}\bar{F}_{ij}$. Further using Eqs. (A7), (A8), and (A9), we get

$$\begin{aligned} &\frac{a(N-1)}{2(N-2+a^2)}\,\frac{1}{V}\partial^2 V + 4\pi a \left[e^{-\frac{2a}{N-1}\phi} U^2 \varphi^* (P_0 + e_0 A_0)^2 \varphi + e^{\frac{2a}{N-1}\phi} m^2\, \varphi^* \varphi\right] U^{\frac{2}{N-2}} \\ &= \frac{a}{2}\, e^{-\frac{4a}{N-1}\phi} U^{-\frac{2}{N-2}} \bar{F}^2 + 4\pi a U^{-\frac{2}{N-2}} \varphi^* (P_i + e_0 \hat{A}_i)^2 \varphi\,. \end{aligned} \tag{A19}$$

Finally we use (A12) and rearrange the equation, and because the right hand side of Eq. (A19) is $O(v^2)$, we find that the dilaton equation in the lowest order can be reduced to

$$\partial^2 V + 16\pi\,\frac{N-2+a^2}{N-1}\,m^2 U^{\frac{N}{N-2}}|\varphi|^2 = 0\,. \tag{A20}$$

The time-time component of the gravitational field equation (A4) can be treated in the same manner. For the first step, we use the metric ansatz (A6) and then get

$$\begin{aligned}
&\frac{N-1}{N-2}\,U^{-\frac{2}{N-2}}\partial_\ell\Big(\frac{\partial_\ell U}{U}\Big) + \frac{1}{2}\frac{N-1}{N-2}\,U^{-\frac{2}{N-2}}\Big(\frac{\partial_\ell U}{U}\Big)^2 - \frac{3}{8}\,U^{-2}U^{-\frac{4}{N-2}}G^2 \\
&= \frac{4}{N-1}\Big[-\frac{1}{2}\,U^{-\frac{2}{N-2}}(\partial_k\phi)^2\Big] + e^{-\frac{4a}{N-1}\phi}\Big[-U^2U^{-\frac{2}{N-2}}(F_{0k})^2 - \frac{1}{2}U^{-\frac{4}{N-2}}\bar{F}^2\Big] \\
&\quad + 8\pi\Big\{e^{-\frac{2a}{N-1}\phi}\Big[-U^2\varphi^*(P_0+e_0A_0)^2\,\varphi - U^{-\frac{2}{N-2}}\varphi^*(P_i+e_0\hat{A}_i)^2\varphi\Big] - e^{\frac{2a}{N-1}\phi}m^2\varphi^*\varphi\Big\}\,,
\end{aligned} \tag{A21}$$

where $G^2 = G_{ij}G_{ij}$. Next, we use V and obtain

$$\begin{aligned}
&\frac{N-1}{N-2+a^2}\frac{1}{V}\,\partial^2 V + \,8\pi\Big[e^{-\frac{2a}{N-1}\phi}U^2\varphi^*(P_0+e_0A_0)^2\,\varphi + e^{\frac{2a}{N-1}\phi}m^2\varphi^*\varphi\Big]U^{\frac{2}{N-2}} \\
&= \frac{3}{8}\,U^{-2}U^{-\frac{2}{N-2}}G^2 - \frac{1}{2}e^{-\frac{4a}{N-1}}\bar{F}^2 - 8\pi e^{-\frac{2a}{N-1}\phi}\varphi^*(P_i+e_0\hat{A}_i)^2\varphi\,.
\end{aligned} \tag{A22}$$

Finally we pick up a part of the lowest order. The reduced equation is the same as Eq. (A20).

The temporal component of the electromagnetic field equation (A5), in the lowest order, can be read as

$$\begin{aligned}
&U^{-\frac{2}{N-2}}\partial_k\Big[e^{-\frac{4a}{N-1}\phi}\Big(U^2F_{0k} + U^{-\frac{2}{N-2}}B_\ell\bar{F}_{\ell k}\Big)\Big] \\
&= -8\pi e_0\,e^{-\frac{2a}{N-1}\phi}\Big[U^2\varphi^*(P_0+e_0A_0)\,\varphi + U^{-\frac{2}{N-2}}B_k\,\varphi^*(P_k+e_0\hat{A}_k)\,\varphi\Big]\,.
\end{aligned} \tag{A23}$$

This equation is equivalent to

$$\begin{aligned}
&\sqrt{\frac{N-1}{2(N-2+a^2)}}\,\partial^2 V - 8\pi e_0\,e^{-\frac{2a}{N-1}\phi}U^{\frac{N}{N-2}}U\varphi^*(P_0+e_0A_0)\,\varphi \\
&= \partial_k\Big[\,e^{-\frac{4a}{N-1}\phi}U^{-\frac{2}{N-2}}B_\ell\bar{F}_{\ell k}\Big] + 8\pi e_0\,e^{-\frac{2a}{N-1}\phi}B_k\,\varphi^*(P_k+e_0\hat{A}_k)\,\varphi\,.
\end{aligned} \tag{A24}$$

The right hand side of this equation is $O(v^2)$. Together with Eq. (A11), we find that Eq. (A24) reduces to Eq. (A20) in the lowest order in v.

From the time-space component of the gravitational field equation (A4), we obtain

$$\begin{aligned}
&-\frac{1}{2}\,UU^{-\frac{1}{N-2}}\partial_\ell\Big(U^{-2}U^{-\frac{2}{N-2}}G_{\ell i}\Big) \\
&= -\,2e^{-\frac{4a}{N-1}\phi}UU^{-\frac{3}{N-2}}F_{0k}\bar{F}_{ik} - 16\pi e^{-\frac{2a}{N-1}\phi}UU^{-\frac{1}{N-2}}\Big[\,\varphi^*(P_0+e_0A_0)(P_i+e_0\hat{A}_i)\varphi\Big]\,.
\end{aligned} \tag{A25}$$

This can be reduced to

$$
\begin{aligned}
&\partial_\ell\Big[V^{\frac{2(a^2-1)}{N-2+a^2}}\,\frac{1}{V^2}\,G_{\ell i}\Big]\\
&= -4\sqrt{\frac{N-1}{2(N-2+a^2)}}\,\partial_k\Big[V^{\frac{2(a^2-1)}{N-2+a^2}}\,\frac{1}{V}\bar{F}_{ki}\Big] + 4\sqrt{\frac{N-1}{2(N-2+a^2)}}\frac{1}{V}\,\partial_k\Big[V^{\frac{2(a^2-1)}{N-2+a^2}}\,\bar{F}_{ki}\Big]\\
&\quad +32\pi e^{-\frac{2a}{N-1}\phi}\Big[\,\varphi^*(P_0+e_0A_0)(P_i+e_0\hat{A}_i)\varphi\Big]\\
&\approx -4\sqrt{\frac{N-1}{2(N-2+a^2)}}\,\partial_k\Big[V^{\frac{2(a^2-1)}{N-2+a^2}}\,\frac{1}{V}\bar{F}_{ki}\Big] + 4\sqrt{\frac{N-1}{2(N-2+a^2)}}\frac{1}{V}\,\partial_k\Big[V^{\frac{2(a^2-1)}{N-2+a^2}}\,\bar{F}_{ki}\Big]\\
&\quad -32\pi m\frac{1}{V}e^{-\frac{2a}{N-1}\phi}\varphi^*(P_i+e_0\hat{A}_i)\varphi\,. \qquad \text{(A26)}
\end{aligned}
$$

On the other hand, the spatial component of the electromagnetic field equation (A5) reads

$$
\partial_k\Big[\,e^{-\frac{4a}{N-1}\phi}U^{-\frac{2}{N-2}}\bar{F}_{ki}\Big] = 8\pi e_0\,e^{-\frac{2a}{N-1}\phi}\varphi^*(P_k+e_0\hat{A}_k)\varphi\,, \qquad \text{(A27)}
$$

or equivalently,

$$
\partial_k\Big[\,V^{\frac{2(a^2-1)}{N-2+a^2}}\,\bar{F}_{ki}\Big] = 8\pi e_0\,e^{-\frac{2a}{N-1}\phi}\varphi^*(P_k+e_0\hat{A}_k)\varphi\,. \qquad \text{(A28)}
$$

Finally using the mass-charge relation (A11), we have

$$
\sqrt{\frac{N-1}{2(N-2+a^2)}}\,\partial_k\Big[\,V^{\frac{2(a^2-1)}{N-2+a^2}}\,\bar{F}_{ki}\Big] = 8\pi m\,e^{-\frac{2a}{N-1}\phi}\varphi^*(P_k+e_0\hat{A}_k)\varphi\,. \qquad \text{(A29)}
$$

By taking the same low-energy approximation into the total action (A2), we obtain the effective lagrangian density $\mathcal{L}$, where

$$
S = \int d^{N+1}x\;\mathcal{L}\,. \qquad \text{(A30)}
$$

Note that:

$$
\begin{aligned}
R &= -\frac{2}{N-2}\,U^{-\frac{2}{N-2}}\partial_\ell\Big(\frac{\partial_\ell U}{U}\Big) - \frac{N-1}{N-2}\,U^{-\frac{2}{N-2}}\Big(\frac{\partial_\ell U}{U}\Big)^2 + \frac{1}{4}\,U^{-2}U^{-\frac{4}{N-2}}G^2\\
&= -\frac{2}{N-2}\,U^{-\frac{2}{N-2}}\partial_\ell\Big(\frac{\partial_\ell U}{U}\Big) - \frac{(N-1)(N-2)}{(N-2+a^2)^2}\,U^{-\frac{2}{N-2}}\Big(\frac{\partial_\ell V}{V}\Big)^2\\
&\quad +\frac{1}{4}V^{\frac{2(a^2-1)}{N-2+a^2}}U^{-\frac{2}{N-2}}\frac{1}{V^2}G^2\,, \qquad \text{(A31)}
\end{aligned}
$$

$$
\begin{aligned}
-\frac{4}{N-1}\nabla_\mu\phi\nabla^\mu\phi &= -\frac{4}{N-1}U^{-\frac{2}{N-2}}(\partial_\ell\phi)^2\\
&= -\frac{(N-1)a^2}{(N-2+a^2)^2}U^{-\frac{2}{N-2}}\Big(\frac{\partial_\ell V}{V}\Big)^2\,, \qquad \text{(A32)}
\end{aligned}
$$

$$-e^{-\frac{4a}{N-1}\phi}F^2 = V^{\frac{2a^2}{N-2+a^2}}\left[2U^2U^{-\frac{2}{N-2}}(F_{0i})^2 - U^{-\frac{4}{N-2}}\bar{F}^2\right]$$
$$= \frac{N-1}{N-2+a^2}U^{-\frac{2}{N-2}}\Big(\frac{\partial_\ell V}{V}\Big)^2 - V^{\frac{2(a^2-1)}{N-2+a^2}}U^{-\frac{2}{N-2}}\bar{F}^2\,. \tag{A33}$$

Now we find:

$$\begin{aligned}
\mathcal{L} &= \sqrt{-g}\,\Big\{\frac{1}{16\pi}\,\Big[R - \frac{4}{N-1}\,(\nabla\phi)^2 - e^{-\frac{4a}{N-1}\phi}\,F^2\Big] \\
&\qquad + \Big[-\varphi^* e^{-\frac{2a}{N-1}\phi}g^{\mu\nu}(P_\mu + e_0A_\mu)(P_\nu + e_0A_\nu)\,\varphi - m^2 e^{\frac{2a}{N-1}\phi}\varphi^*\varphi\Big]\Big\} \\
&\approx \frac{1}{16\pi}\,V^{\frac{2(a^2-1)}{N-2+a^2}}\,\Big[\frac{1}{4}\,\frac{1}{V^2}\,G^2 - \bar{F}^2\Big] \\
&\quad + U^{\frac{N}{N-2}}\Big[V\,\varphi^*(P_0 + e_0A_0)^2\varphi - \frac{1}{V}m^2\varphi^*\varphi - \frac{1}{V}\,V^{\frac{2(a^2-1)}{N-2+a^2}}\,\varphi^*(P_i + e_0\hat{A}_i)^2\varphi\Big] \\
&= \frac{1}{16\pi}\,V^{\frac{2(a^2-1)}{N-2+a^2}}\,\Big[\frac{N-2+a^2}{N-a^2}\,\Big(\hat{F}^2 - \frac{1}{4}\,H^2\Big)\Big] \\
&\quad + U^{\frac{N}{N-2}}\Big[V\varphi^*(P_0 + e_0A_0)^2\varphi - \frac{1}{V}m^2\varphi^*\varphi - \frac{1}{V}\,V^{\frac{2(a^2-1)}{N-2+a^2}}\,\varphi^*(P_i + e_0\hat{A}_i)^2\varphi\Big]\,.
\end{aligned} \tag{A34}$$

Here we have defined an antisymmetric tensor field H_{ij} as

$$H_{ij} \equiv 4\,\sqrt{\frac{N-1}{2(N-2+a^2)}}\bar{F}_{ij} + \frac{1}{V}\,G_{ij}\,. \tag{A35}$$

H_{ij} does not couple to the scalar field φ, thus we set $H_{ij} \equiv 0$. Then both Eqs. (A26) and (A29) can be read as

$$-\,\frac{N-2+a^2}{N-a^2}\,\partial_\ell\Big[V^{\frac{2(a^2-1)}{N-2+a^2}}\,\hat{F}_{\ell i}\Big] = 8\pi e_0\,e^{-\frac{2a}{N-1}\phi}\,\varphi^*(P_i + e_0\hat{A}_i)\,\varphi\,. \tag{A36}$$

To proceed further, we introduce a non-relativistic field ψ:

$$\psi \equiv \sqrt{2m}\,U^{\frac{N}{2(N-2)}}\,\varphi\,, \tag{A37}$$

where since the spatial volume measure $(g^{(N)})^{1/4} = U^{\frac{N}{2(N-2)}}$, we obtain a correct measure for a usual spatial volume.

Finally we get the effective lagrangian density in the low energy limit:

$$\begin{aligned}
\mathcal{L} &= \psi^*\,(-\,P_0 - m)\psi - \frac{1}{2m\,V^{(N-a^2)/(N-2+a^2)}}\,\psi^*\,(\boldsymbol{P} + e_0\hat{\boldsymbol{A}})^2\psi \\
&\qquad + \frac{1}{16\pi}\,\frac{N-2+a^2}{N-a^2}\,\frac{1}{V^{2(1-a^2)/(N-2+a^2)}}\,\hat{F}^2 \qquad (a^2 \neq N)\,,
\end{aligned} \tag{A38}$$

where V satisfies the following equation:

$$\partial^2 V + 8\pi \frac{N-2+a^2}{N-1} m\, |\psi|^2 = 0\,. \tag{A39}$$

Varying this effective lagrangian (A38) with respect to $\hat{\boldsymbol{A}}$, we can derive again the field equation equivalent to Eq. (A36) in the low-energy approximation.

For $a^2 = N$, since the scalar field does not couple to the vector field, the effective lagrangian density at the lowest order is

$$\mathcal{L} = \psi^* \left(-P_0 - m\right)\psi - \frac{1}{2m} \psi^* \boldsymbol{P}^2 \psi \qquad (a^2 = N)\,. \tag{A40}$$

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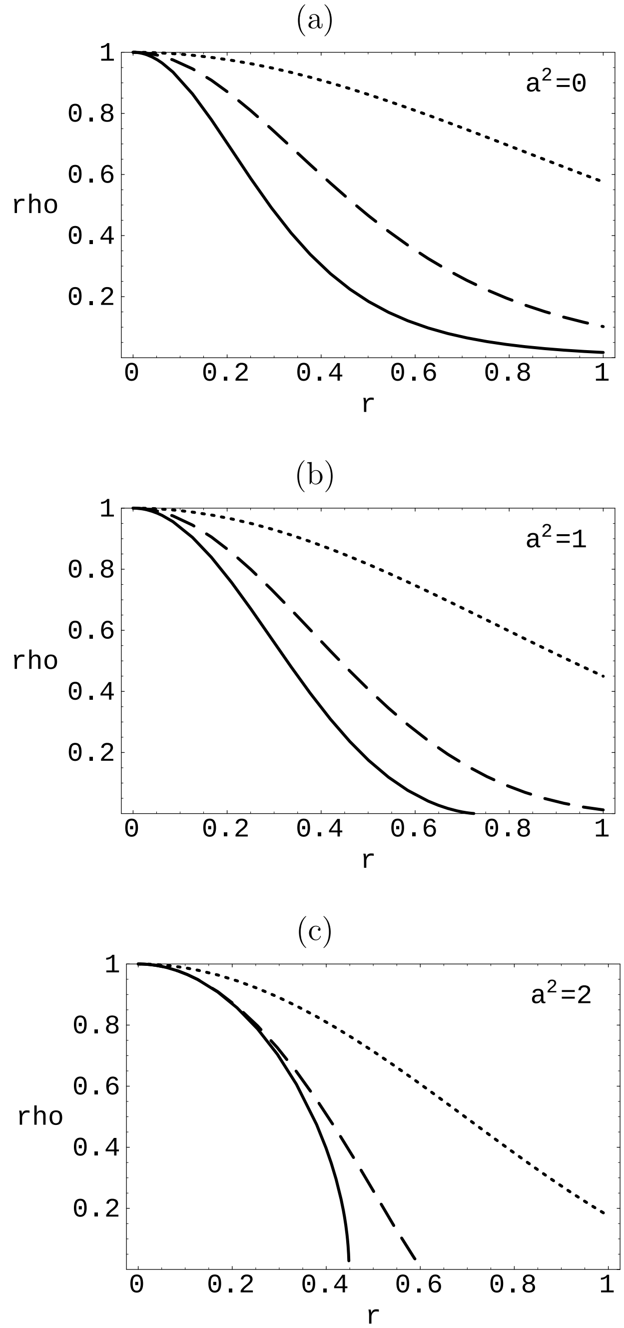

FIG. 1. The density distribution of the isothermal sphere of “extreme black holes” for different values of the coupling constant of the dilaton field; (a) $a^2 = 0$, (b) $a^2 = 1$ and (c) $a^2 = 2$. The solid line denotes $\delta = 0$ (the high temperature limit), the broken line $\delta = 1$ and the dotted line $\delta = 10$.